Analytic and Geometric Inequalities and Applications

Mathematics and Its Applications

Managing Editor:

M. HAZEWINKEL

Centre for Mathematics and Computer Science, Amsterdam, The Netherlands

Volume 478

Analytic and Geometric Inequalities and Applications

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SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-94-010-5938-1   ISBN 978-94-011-4577-0 (eBook)
DOI 10.1007/978-94-011-4577-0

Printed on acid-free paper

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PREFACE

Analytic and Geometric Inequalities and Applications is devoted to recent advances in a variety of inequalities of Mathematical Analysis and Geometry. Subjects dealt with in this volume include: Fractional order inequalities of Hardy type, differential and integral inequalities with initial time difference, multi-dimensional integral inequalities, Opial type inequalities, Gruss' inequality, Furuta inequality, Laguerre–Samuelson inequality with extensions and applications in statistics and matrix theory, distortion inequalities for analytic and univalent functions associated with certain fractional calculus and other linear operators, problem of infimum in the positive cone, alpha-quasi-convex functions defined by convolution with incomplete beta functions, Chebyshev polynomials with integer coefficients, extremal problems for polynomials, Bernstein's inequality and Gauss–Lucas theorem, numerical radii of some companion matrices and bounds for the zeros of polynomials, degree of convergence for a class of linear operators, open problems on eigenvalues of the Laplacian, fourth order obstacle boundary value problems, bounds on entropy measures for mixed populations as well as controlling the velocity of Brownian motion by its terminal value. A wealth of applications of the above is also included.

We wish to express our appreciation to the distinguished mathematicians who contributed to this volume. Finally, it is our pleasure to acknowledge the fine cooperation and assistance provided by the staff of Kluwer Academic Publishers.

June 1999

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PROBLEM OF INFIMUM IN THE POSITIVE CONE

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Abstract. It is known that for (bounded) self-adjoint operators A, B on a Hilbert space $\mathcal{H}$ the infimum $A \wedge B$, with respect to the order induced by the cone of positive (semi-definite) operators, exists only when A and B are comparable, that is, $A \geq B$ or $A \leq B$. In this paper we present a necessary and sufficient condition for that, given $A, B \geq 0$, the infimum considered in the positive cone exists.

1. Problem

The space of (bounded) self-adjoint operators on a Hilbert space $\mathcal{H}$ is provided with the order relation induced by positive semi-definiteness; for self-adjoint A, B the order relation $A \geq B$ means $A - B$ is *positive* (*semi-definite*). In particular, $A \geq 0$ means positive semi-definiteness of A. With respect to this order, the space of self-adjoint operators does not become a lattice. More exactly, for self-adjoint A, B the infimum $A \wedge B$ exists only when A and B is comparable, that is, $A \geq B$ or $A \leq B$.

The situation is different if we take the *positive cone* $\mathcal{P} = \mathcal{P}(\mathcal{H})$, the cone of positive (semi-definite) operators, in place of the whole space of self-adjoint operators. For instance, it is well known that for any two orthoprojections P, Q the infimum $P \wedge Q$ in $\mathcal{P}$ always exists and is equal to the orthoprojection to the intersection of the range of P and that of Q.

In this paper we present a necessary and sufficient condition for that, given $A, B \geq 0$, the infimum $A \wedge B$ in $\mathcal{P}$ exists. This problem has been studied by several authors in mathematical physics (see [6], [7], [8] and [9]). Among others, Moreland and Gudder [9] have solved this problem in the finite dimensional case. We explain also how our result reduces to theirs in the finite dimensional case.

1991 *Mathematics Subject Classification.* Primary 47D20; Secondary 47A63
Key words and phrases. Positive cone; Lattice, Operator inequalities

T.M. Rassias and H.M. Srivastava (eds.), Analytic and Geometric Inequalities and Applications, 1–12.
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2. Reduction

To discuss existence of the infimum $A \wedge B$ in $\mathcal{P}$, we may assume $\ker(A+B) = \{0\}$. (Here ker denotes the *kernel.*) In fact, let P be the orthoprojection to the orthocomplement $\mathcal{K}$ of $\ker(A+B)$. Then the map

$$0 \leq X \longmapsto \Phi(X) = PX|_{\mathcal{K}}$$

gives an affine, order-isomorphism from $\{X \ ; \ 0 \leq X \leq A, B\}$ onto a subset $\mathcal{Q}$ of the positive cone $\mathcal{P}(\mathcal{K})$ satisfying that $Y \in \mathcal{Q}, 0 \leq Z \leq Y \implies Z \in \mathcal{Q}$, and $\ker(\Phi(A) + \Phi(B)) = \{0\}$.

When $\ker(A+B) = \{0\}$, there is an affine, order-isomorphism φ from the set $\{X \ ; \ 0 \leq X \leq A+B\}$ onto the set of positive contractions, that is, $\{Y \ ; \ 0 \leq Y \leq 1\}$, given by

$$X = (A+B)^{1/2} \cdot \varphi(X) \cdot (A+B)^{1/2}. \tag{1}$$

Here important is the fact that

$$\varphi(A) + \varphi(B) = 1,$$

in particular, $\varphi(A)$ and $\varphi(B)$ commute. Therefore we first discuss existence of the infimum $A \wedge B$ in $\mathcal{P}$ under the assumption $A + B = 1$.

3. Commuting Case

Suppose that

$$A, \ B \geq 0, \quad A + B = 1. \tag{2}$$

Then according to the spectral theorem there is uniquely an increasing family of orthoprojection $\{E(\lambda) \ ; \ 0 \leq \lambda \leq 1\}$ with $E(1) = 1$ such that

$$A = \int_0^1 \lambda dE(\lambda) \quad \text{and} \quad B = \int_0^1 (1-\lambda) dE(\lambda). \tag{3}$$

Lemma 1. *Under the setting* (3), *if the infimum* $A \wedge B$ *in* $\mathcal{P}$ *exists, then*

$$A \wedge B = \int_0^1 \min(\lambda, 1-\lambda) dE(\lambda).$$

Proof. Let

$$C \stackrel{\text{def}}{=} \int_0^1 \min(\lambda, 1-\lambda) dE(\lambda).$$

Then obviously $0 \leq C \leq A, \ B$. We claim that C is a *maximal* element in the set $\{X \ ; \ 0 \leq X \leq A, B\}$ (cf. Ando [5]). To see this, take D such that

$$C \leq D \leq A, \ B.$$

Then

$$0 \le D - C \le A - C,\ B - C$$

so that there exist contractions X, Y such that

$$(D-C)^{1/2} = (A-C)^{1/2}X \quad \text{and} \quad (D-C)^{1/2} = (B-C)^{1/2}Y,$$

which implies

$$\operatorname{ran}(D-C)^{1/2} \subset \operatorname{ran}(A-C)^{1/2} \cap \operatorname{ran}(B-C)^{1/2}.$$

Here ran denotes the *range.* Since by the spectral representation (3)

$$\begin{aligned}(A-C)^{1/2} &= \int_0^1 \sqrt{\max(2\lambda-1,0)}dE(\lambda) = \int_{\frac{1}{2}+}^1 \sqrt{2\lambda-1}dE(\lambda),\\ (B-C)^{1/2} &= \int_0^1 \sqrt{\max(1-2\lambda,0)}dE(\lambda) = \int_0^{\frac{1}{2}-} \sqrt{1-2\lambda}dE(\lambda),\end{aligned}$$

we can infer that

$$\operatorname{ran}(A-C)^{1/2} \cap \operatorname{ran}(B-C)^{1/2} = \{0\},$$

which implies

$$\operatorname{ran}(D-C)^{1/2} = \{0\},$$

or equivalently $D - C = 0$, that is, $D = C$. This proves our claim on maximality. Now if the infimum $A \wedge B$ in $\mathcal{P}$ exists, it must coincide with the maximial element C. This completes the proof. □

Theorem 2. *Under the setting* (3), *the infimum* $A \wedge B$ *in* $\mathcal{P}$ *exists if and only if*

$$\int_{0+}^{1-} \lambda dE(\lambda) \ge \int_{0+}^{1-} (1-\lambda)dE(\lambda)$$

or

$$\int_{0+}^{1-} \lambda dE(\lambda) \le \int_{0+}^{1-} (1-\lambda)dE(\lambda).$$

In each case, the infimum coincides with the smaller of the above two and is equal to

$$\int_0^1 \min(\lambda, 1-\lambda)dE(\lambda).$$

Here remark

$$\int_{0+}^{1-} \lambda dE(\lambda) = A \cdot \{1 - E(\{0\}) - E(\{1\})\}, \tag{4}$$

$$\int_{0+}^{1-} (1-\lambda)dE(\lambda) = B \cdot \{1 - E(\{0\}) - E(\{1\})\}. \tag{5}$$

Proof. A proof of the "if" part is easy. Suppose, for instance, via (4) and (5), that

$$A \cdot \{1 - E(\{0\}) - E(\{1\})\} \ \geq\ B \cdot \{1 - E(\{0\}) - E(\{1\})\},$$

and let

$$C_0 \stackrel{\text{def}}{=} B \cdot \{1 - E(\{0\}) - E(\{1\})\}.$$

Then obviously $0 \leq C_0 \leq A,\ B$. Take D such that $0 \leq D \leq A,\ B$. Then since by (3)

$$A \cdot E(\{0\}) = 0 = B \cdot E(\{1\}),$$

the inequality $0 \leq D \leq A,\ B$ implies

$$D \cdot E(\{0\}) = D \cdot E(\{1\}) = 0$$

so that

$$\begin{aligned} D &= \{1 - E(\{0\}) - E(\{1\})\} \cdot D \cdot \{1 - E(\{0\}) - E(\{1\})\} \\ &\leq \{1 - E(\{0\}) - E(\{1\})\} \cdot B \cdot \{1 - E(\{0\}) - E(\{1\})\} = C_0. \end{aligned}$$

This proves that C_0 is the infimumu of A, B in $\mathcal{P}$.

To prove the "only if" part by contradiction, suppose that

$$\int_{0+}^{1-} \lambda dE(\lambda) \not\geq \int_{0+}^{1-} (1 - \lambda) dE(\lambda) \tag{6}$$

and

$$\int_{0+}^{1-} \lambda dE(\lambda) \not\leq \int_{0+}^{1-} (1 - \lambda) dE(\lambda), \tag{7}$$

but $A,\ B$ admits the infimum in $\mathcal{P}$. Then by Lemma 1 this infimum coincides with

$$C \stackrel{\text{def}}{=} \int_{0}^{1} \min(\lambda, 1 - \lambda) dE(\lambda).$$

By assumptions (6) and (7), there exist disjoint subsets $\Delta_1, \Delta_2 \subset (0, 1)$ and $\epsilon > 0$ such that $E(\Delta_1) \neq 0, E(\Delta_2) \neq 0$ and

$$1 - 3\epsilon \ \geq\ \lambda \geq (1 - \lambda) + \epsilon \qquad (\lambda \in \Delta_1), \tag{8}$$

$$1 - 3\epsilon \ \geq\ 1 - \lambda \geq \lambda + \epsilon \qquad (\lambda \in \Delta_2). \tag{9}$$

We may assume, without loss of generality, that

$$\dim(\mathrm{ran} E(\Delta_1)) \geq \dim(\mathrm{ran} E(\Delta_2)).$$

Then there is a partial isometry V with domain $\mathrm{ran}E(\Delta_2)$ and range in $\mathrm{ran}E(\Delta_1)$. Define an operator D by

$$\begin{aligned} D \stackrel{\text{def}}{=} & B \cdot E(\Delta_1) - \epsilon E(\Delta_1) + A \cdot E(\Delta_2) \\ & -\epsilon E(\Delta_2) + \sqrt{2}\epsilon V \cdot E(\Delta_2) + \sqrt{2}\epsilon E(\Delta_2) \cdot V^*. \end{aligned}$$

We claim that $A,\ B \geq D \geq 0$. First let us show $D \geq 0$. Since by (8)

$$1-\lambda \geq 3\epsilon \quad (\lambda \in \Delta_1)$$

it follows from (3) that

$$B \cdot E(\Delta_1) - \epsilon E(\Delta_1) \geq 2\epsilon E(\Delta_1),$$

and similarly from (9) and (3) that

$$A \cdot E(\Delta_2) - \epsilon E(\Delta_2) \geq 2\epsilon E(\Delta_2).$$

Further since by definition of V

$$E(\Delta_2) \cdot V^*V \cdot E(\Delta_2) = E(\Delta_2) \quad \text{and} \quad E(\Delta_1) \cdot V \cdot E(\Delta_2) = V \cdot E(\Delta_2)$$

we can conclude from the above that

$$\begin{aligned} D &\geq \sqrt{2}\epsilon\{E(\Delta_1) + E(\Delta_2) + V \cdot E(\Delta_2) + E(\Delta_2) \cdot V^*\} \\ &= \sqrt{2}\epsilon\{E(\Delta_1) + V \cdot E(\Delta_2)\}^*\{E(\Delta_1) + V \cdot E(\Delta_2)\} \geq 0 \end{aligned}$$

Next let us turn to the proof of $A,\ B \geq D$. Since by (3) and (8)

$$(A-B) \cdot E(\Delta_1) \geq \epsilon E(\Delta_1)$$

it follows from the definitin of D and the property of V that

$$\begin{aligned} A - D &= A \cdot E((\Delta_1 \cup \Delta_2)^c) + (A-B) \cdot E(\Delta_1) + \epsilon E(\Delta_1) \\ &\qquad + \epsilon E(\Delta_2) - \sqrt{2}\epsilon V \cdot E(\Delta_2) - \sqrt{2}\epsilon E(\Delta_2) \cdot V^* \\ &\geq \epsilon\{2E(\Delta_1) + E(\Delta_2) - \sqrt{2}V \cdot E(\Delta_2) - \sqrt{2}E(\Delta_2) \cdot V^*\} \\ &= \epsilon\{\sqrt{2}E(\Delta_1) - V \cdot E(\Delta_2)\}^*\{\sqrt{2}E(\Delta_1) - V \cdot E(\Delta_2)\} \geq 0. \end{aligned}$$

This proves $A \geq D$. The inequality $B \geq D$ is proved similarly.

Now since by Lemma 1 C is the infimum $A \wedge B$ in $\mathcal{P}$, we can conclude $C \geq D$. To see that this causes a contradiction, take a unit vector $x \in \mathrm{ran}E(\Delta_2)$ and let $y \stackrel{\text{def}}{=} Vx$. Then y is a unit vector in $\mathrm{ran}E(\Delta_1)$. On the other hand, since by the definitions of C and D

$$C - D = C \cdot E((\Delta_1 \cup \Delta_2)^c) + \epsilon\{E(\Delta_1) + E(\Delta_2) - \sqrt{2}V \cdot E(\Delta_2) - \sqrt{2}E(\Delta_2) \cdot V^*\},$$

we have

$$\begin{aligned} \langle (C-D)(x+y), x+y \rangle &= \epsilon\{\langle x,x\rangle + \langle y,y\rangle - \sqrt{2}\langle y,y\rangle - \sqrt{2}\langle x,x\rangle\} \\ &= -2(\sqrt{2}-1)\epsilon < 0, \end{aligned}$$

which contradicts $C - D \geq 0$. This proves the theorem. $\square$

4. Parallel sum and short

How can we recapture

$$\int_{0+}^{1-} \lambda dE(\lambda) \quad \text{and} \quad \int_{0+}^{1-} (1-\lambda)dE(\lambda)$$

from A and B in the form (3) without appealing to the spectral representation ?

For this purpose, recall the definition of parallel sum operation $X : Y$ for $X, Y \geq 0$, introduced for matrices by Anderson and Duffin [1] and later for Hilbert space operators by Pekarev and Smul'yan [11].

As a positive quadratic form, *parallel sum* $X : Y$ is defined as

$$\langle (X:Y)a, a\rangle \stackrel{\text{def}}{=} \inf\{\langle Xb, b\rangle + \langle Yc, c\rangle \ ; \ b + c = a\}. \tag{10}$$

When $X + Y$ is invertible, this definition takes the form

$$X : Y = X - X(X+Y)^{-1}X = Y - Y(X+Y)^{-1}Y, \tag{11}$$

and when both X and Y are invertible,

$$X : Y = (X^{-1} + Y^{-1})^{-1}. \tag{12}$$

As a consequence of definition (10), parallel addition has the following properties; here X, Y, Z are all in $\mathcal{P}$,

(a) $0 \leq X : Y = Y : X \leq X,\ Y,$

(b) $(\alpha X) : (\alpha Y) = \alpha(X : Y) \quad (\alpha \geq 0),$

(c) $(X : Y) : Z = X : (Y : Z),$

(d) the map $(X, Y) \longmapsto X : Y$ is (jointly) *monotone* in the sense

$$X_1 \geq X_2,\ Y_1 \geq Y_2 \implies X_1 : Y_1 \geq X_2 : Y_2$$

(e) the map is (strongly) *continuous from above* in the sense:

$$X_n \downarrow X,\ Y_n \downarrow Y \implies X_n : Y_n \downarrow X : Y.$$

A little non-trivial is the following relation for ranges;

(f) $$\operatorname{ran}(A : B)^{1/2} = \operatorname{ran}(A^{1/2}) \cap \operatorname{ran}(B^{1/2}).$$

Recall the definition of short (operation) (see Anderson and Trapp [2]). Given a closed subspace $\mathcal{M} \subset \mathcal{H}$, for each $X \geq 0$ there is the maximum element in the set

$$\{Z \ ; \ 0 \leq Z \leq X,\ \operatorname{ran}(Z) \subset \mathcal{M}\}.$$

This maximum element is called the *short* of X to the subspace $\mathcal{M}$. By identifying a closed subspace $\mathcal{M}$ with the orthoprojection $P = P_{\mathcal{M}}$ to itself, we shall denote the short by $[P]X$. Since inclusion $\mathrm{ran}(Z) \subset \mathcal{M}$ is equivalent to the requirement that there is $\gamma = \gamma(Z) \geq 0$ such that $Z \leq \gamma P$, the short $[P]X$ is defined by

$$[P]X = \max\{Z \ ; \ 0 \leq Z \leq X, \ \gamma P \text{ for some } \gamma \geq 0\} \tag{13}$$

Therefore if P commutes with X, then $[P]X = XP$. In general case, it is not difficult to see

$$[P]X = \lim_{n\to\infty}(nP) : X \quad \text{(strong convergence)}.$$

Motivated by this relation, Ando [4] introduced a notion of generalized short $[Y]X$ with $Y \geq 0$ (called the *Y-absolutely continuous part* of X) by

$$[Y]X \stackrel{\text{def}}{=} \lim_{n\to\infty}(nY) : X. \tag{14}$$

Here strong convergence on the right side is always guaranted because

$$0 \leq (nY) : X \leq ((n+1)Y) : X \leq X \quad (n = 1, 2, \ldots).$$

The following properties of generalized short are immediate; here all X, Y are in $\mathcal{P}$,

(A) $\quad 0 \leq [Y]X \leq X \quad \text{and} \quad [Y]Y = Y,$

(B) $\quad [Y](\alpha X) = \alpha[Y]X \quad (\alpha \geq 0),$

(C) $\quad \alpha Z \leq Y \leq \beta Z$ for some $\alpha, \beta > 0 \implies [Y]X = [Z]X,$

(D) $\quad [Y]X = [X : Y]X,$

(E) $\quad [P]1 = P$ for orthoprojection P,

(F) $\quad$ the map $(X, Y) \longmapsto [Y]X$ is (jointly) *monotone* in the sense:

$$X_1 \geq X_2, \ Y_1 \geq Y_2 \implies [Y_1]X_1 \geq [Y_2]X_2.$$

A little non-trivial is the following property whose proof is found in Nishio [8]:

(G) $\quad [Y]X = [[X]Y]X.$

Lemma 3. *Under the setting* (3)

$$\int_{0+}^{1-} \lambda dE(\lambda) = [B]A \quad \textit{and} \quad \int_{0+}^{1-} (1-\lambda) dE(\lambda) = [A]B.$$

Proof. By (11)

$$(nB) : A = \int_0^1 f_n(\lambda) dE(\lambda)$$

where

$$f_n(\lambda) \stackrel{def}{=} \lambda - \frac{\lambda^2}{\lambda + n(1-\lambda)}.$$

As $n \to \infty$, $f_n(\lambda)$ converges increasingly to λ for $0 < \lambda < 1$ while

$$f_n(1) = f_n(0) = 0 \quad (n = 1, 2, \ldots).$$

Therefore by definition (14)

$$[B]A = \lim_{n\to\infty} \int_0^1 f_n(\lambda) dE(\lambda) = \int_0^1 \lim_{n\to\infty} f_n(\lambda) dE(\lambda) = \int_{0+}^{1-} \lambda dE(\lambda),$$

and similarly

$$[A]B = \int_{0+}^{1-} (1-\lambda) dE(\lambda).$$

This completes the proof. □

Now the content of Theorem 2 can be stated without use of spectral representation.

Theorem 4. *When $A, B \geq 0$ and $A + B = 1$, the infimum $A \wedge B$ in $\mathcal{P}$ exists if and only if $[A]B$ and $[B]A$ are comparable. In this case*

$$A \wedge B = \min([A]B, [B]A).$$

5. General case

Fix $A,\ B \geq 0$ with $\ker(A+B) = \{0\}$. By (1) there is an affine, order isomorphism $\varphi(\cdot)$ from the set $\{X\ ;\ 0 \leq X \leq A+B\}$ onto the set $\{Z\ ;\ 0 \leq Z \leq 1\}$, for which $\varphi(A) + \varphi(B) = 1$.

Lemma 5. *The map φ in* (1) *satisfies the condition*

$$\varphi([Y]X) = [\varphi(Y)]\varphi(X) \quad (0 \leq X,\ Y \leq A+B).$$

Proof. First let us establish the following identity

$$\varphi(X : Y) = \varphi(X) : \varphi(Y). \tag{15}$$

Take an arbitrary vector $a \in \mathcal{H}$. Since $\ker(A+B) = \{0\}$ implies that $\operatorname{ran}(A+B)^{1/2}$ is dense in $\mathcal{H}$, by (10) and (1)

$$\begin{aligned}
&\langle(\varphi(X) : \varphi(Y))(A+B)^{1/2}a, (A+B)^{1/2}a\rangle \\
&\quad = \inf\{\langle\varphi(X)b', b'\rangle + \langle\varphi(Y)c', c'\rangle\ ;\ (A+B)^{1/2}a = b' + c'\} \\
&\quad = \inf\{\langle\varphi(X)(A+B)^{1/2}b, (A+B)^{1/2}b\rangle + \langle\varphi(Y)(A+B)^{1/2}c, (A+B)^{1/2}c\rangle\ ; \\
&\qquad\qquad a = b + c\} \\
&\quad = \inf\{\langle Xb, b\rangle + \langle Yc, c\rangle\ ;\ a = b + c\} = \langle(X : Y)a, a\rangle \\
&\quad = \langle(A+B)^{1/2}\varphi(X : Y)(A+B)^{1/2}a, a\rangle \\
&\quad = \langle\varphi(X : Y)(A+B)^{1/2}a, (A+B)^{1/2}a\rangle,
\end{aligned}$$

which yields (15), because $(A+B)^{1/2}$ has dense range.

Again take an arbitrary vector $a \in \mathcal{H}$. Then by definition (1), (14) and (15)

$$\begin{aligned}\langle \varphi([Y]X) \cdot (A+B)^{1/2}a, (A+B)^{1/2}a\rangle &= \langle ([Y]X)a, a\rangle \\ = \lim_{n\to\infty} \langle ((nY) : X)a, a\rangle &= \lim_{n\to\infty} \langle \varphi((nY) : X)(A+B)^{1/2}a, (A+B)^{1/2}a\rangle \\ = \lim_{n\to\infty} \langle (n\varphi(Y) : \varphi(X)) \cdot (A+B)^{1/2}a, (A+B)^{1/2}a\rangle \\ = \langle [\varphi(Y)]\varphi(X) \cdot (A+B)^{1/2}a, (A+B)^{1/2}a\rangle,\end{aligned}$$

which yields the assertion because $(A+B)^{1/2}$ has dense range. □

Now we are in position to transform Theorem 4 to the general case.

Theorem 6. *Given $A,\ B \geq 0$, the infimum $A \wedge B$ in $\mathcal{P}$ exists if and only if $[B]A$ and $[A]B$ are comparable, that is,*

$$[B]A \geq [A]B \quad \text{or} \quad [B]A \leq [A]B.$$

In this case

$$A \wedge B = \min([B]A, [A]B).$$

Proof. We may assume that $\ker(A+B) = \{0\}$. Then since the map $\varphi(\cdot)$ in (1) is order-isomorphism between $\{X\ ;\ 0 \leq X \leq A,\ B\}$ and $\{Z\ ;\ 0 \leq Z \leq \varphi(A),\ \varphi(B)\}$, the infimum $A \wedge B$ in $\mathcal{P}$ exists if and only if the infimum $\varphi(A) \wedge \varphi(B)$ in $\mathcal{P}$ exists. Further in this case

$$\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B).$$

By Theorem 4 the infimum $\varphi(A) \wedge \varphi(B)$ in $\mathcal{P}$ exists if and only if $[\varphi(B)]\varphi(A)$ and $[\varphi(A)]\varphi(B)$ are comparable, and in this case the infimum coincides with the smaller one of those two. Then since by Lemma 5

$$[\varphi(B)]\varphi(A) = \varphi([B]A), \quad [\varphi(A)]\varphi(B) = \varphi([A]B)$$

and $\varphi(\cdot)$ is order-isomorphism, the assertion follows. □

6. Special cases

As pointed out in the begining, for every pair of orthoprojections P, Q the infimum $P \wedge Q$ in $\mathcal{P}$ always exists and is equal to the orthoprojection to $\mathrm{ran}(P) \cap \mathrm{ran}(Q)$. Let us discuss a connection of this fact with Theorem 6.

Lemma 7. *Let P be an orthoprojection. Then for every positive contraction $0 \leq A \leq 1$ the infimum $A \wedge P$ in $\mathcal{P}$ exists and is equal to the short $[P]A$.*

Proof. By property (E) and (F), it follows from $0 \leq A \leq 1$ that

$$[P]A \leq [P]1 = P,$$

which implies again by (F) and (G)

$$[P]A = [[P]A]([P]A) \leq [[P]A]P = [A]P.$$

Now the assertion follows from Theorem 6. □

An immediate consequence of Lemma 7 is that for orthoprojections $P,\ Q$

$$[P]Q = [Q]P = P \wedge Q.$$

In this connection recall the identity (see Anderson and Schreiber [6])

$$2(P : Q) = P \wedge Q.$$

Lemma 8. *If $B \geq 0$ is of rank ≤ 1, then for every $A \geq 0$ the infimum $A \wedge B$ in $\mathcal{P}$ exists.*

Proof. $0 \leq [A]B \leq B$ implies that $[A]B$ is a non-negative scalar multiple of B. Similarly each $(nB) : A$ is a non-negative scalar multiple of B $(n = 1, 2, \ldots)$, so is $[B]A$ as the limit. Since any two scalar multiplies of B are comparable, $[B]A$ and $[A]B$ are comparable, and the infimum $A \wedge B$ in $\mathcal{P}$ exists by Theorem 6. □

Theorem 9.

(i) *Let $0 \leq B \leq 1$. Then the infimum $A \wedge B$ in $\mathcal{P}$ exists for every $0 \leq A \leq 1$ if and only if B is of rank ≤ 1 or is an orthoprojection.*

(ii) *Let $B \geq 0$. Then the infimum $A \wedge B$ in $\mathcal{P}$ exists for every $A \geq 0$ if and only if B is of rank ≤ 1.*

Proof. (i) The "if" part is proved in Lemma 7 and Lemma 8. To prove the "only if" part by contradiction, suppose that B is of rank ≥ 2 and is not an orthoprojection. According to the spectral theorem, there are mutually annihilating non-zero orthoprojections $P,\ Q$, commuting with B, such that for some $0 < \epsilon \leq \gamma < 1$

$$BP \geq \gamma P \quad \text{and} \quad \gamma Q \geq BQ \geq \epsilon Q.$$

Let

$$A \stackrel{\text{def}}{=} \frac{\gamma + \epsilon}{2} P + \frac{1 + \gamma}{2} Q.$$

Then $0 \leq A \leq 1$, and A and $BP + BQ$ are not comparable. Since

$$\frac{1 + \gamma}{2\epsilon} B \geq \gamma P + \frac{1 + \gamma}{2} Q \geq A$$

there is $\alpha,\ \beta > 0$ such that $\alpha(A : B) \leq A \leq \beta(A : B)$. Therefore by (A) and (C)

$$[B]A = [B : A]A = [A]A = A.$$

On the other hand, since

$$P + Q \geq A \geq \epsilon(P + Q)$$

and $P+Q$ is an orthoprojection commuting with B, we can see by (C)

$$[A]B = [P+Q]B = BP + BQ.$$

Therefore $[B]A$ and $[A]B$ are not comparable, and by Theorem 6 the infimum $A \wedge B$ in $\mathcal{P}$ does not exist. This proves (i).

(ii) The "if" part is proved in Lemma 8. To prove the "only if" part by contradiction, suppose that B is of rank ≥ 2. Via multiplication by a positive scalar we may assume that $0 \leq B \leq 1/2$. Then since B is not an orthoprojection, as in the proof of (i) there is $0 \leq A \leq 1$ such that the infimum $A \wedge B$ in $\mathcal{P}$ does not exist. This proves (ii). □

7. Finite dimensional case

In this section we assume $\dim(\mathcal{H}) < \infty$. Therefore every $A \geq 0$ has closed range and

$$\operatorname{ran}(A) = \operatorname{ran}(A^{1/2}). \tag{16}$$

We can consider the orthoprojection P_A onto $\operatorname{ran}(A)$. Since there are constants $\alpha,\ \beta > 0$ such that

$$\alpha P_A \leq A \leq \beta P_A,$$

it follows from property (C) for generalized short

$$[A]B = [P_A]B. \tag{17}$$

Let $P_{A,B}$ be the orthoprojection to the intersection $\operatorname{ran}(A) \cap \operatorname{ran}(B)$. Since by (16) and (f)

$$\begin{aligned} \operatorname{ran}(A:B) &= \operatorname{ran}(A:B)^{1/2} = \operatorname{ran}(A^{1/2}) \cap \operatorname{ran}(B^{1/2}) \\ &= \operatorname{ran}(A) \cap \operatorname{ran}(B), \end{aligned}$$

the following relation holds

$$P_{A,B} = P_{A:B}. \tag{18}$$

With a rather complicated method, Moreland and Gudder [9] established the following results (for the finite dimensional case).

(i) *For any orthoprojection P and $0 \leq A \leq 1$ the infimum $A \wedge P$ in $\mathcal{P}$ exists,*

(ii) *for $A, B \geq 0$, the infimum $A \wedge B$ in $\mathcal{P}$ exists if and only if $A \wedge P_{A,B}$ and $B \wedge P_{A,B}$ are comparable. In this case*

$$A \wedge B = \min(A \wedge P_{A,B}, B \wedge P_{A,B}).$$