Die Grundlehren der
mathematischen Wissenschaften in Einzeldarstellungen
Band 162

Rolf Nevanlinna

Analytic Functions

Springer-Verlag Berlin Heidelberg GmbH

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Translated from the Second German Edition by

Phillip Emig

Springer-Verlag Berlin Heidelberg GmbH 1970

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Revised translation of
Eindeutige analytische Funktionen, 2nd ed. 1953
(Grundlehren der mathematischen Wissenschaften, Vol. 46)

ISBN 978-3-642-85592-4 ISBN 978-3-642-85590-0 (eBook)
DOI 10.1007/978-3-642-85590-0

Originally published by Springer-Verlag Berlin Heidelberg New York in 1970.

Library of Congress Catalog Card Number 74-89810
Title No. 5145

Preface

The present monograph on analytic functions coincides to a larg extent with the presentation of the modern theory of single-value analytic functions given in my earlier works "Le théorème de Picarc Borel et la théorie des fonctions méromorphes" (Paris: Gauthier-Villai 1929) and "Eindeutige analytische Funktionen" (Die Grundlehren de mathematischen Wissenschaften in Einzeldarstellungen, Vol. 46, 1s edition Berlin: Springer 1936, 2nd edition Berlin-Göttingen-Heidelberg Springer 1953).

In these presentations I have strived to make the individual result and their proofs readily understandable and to treat them in the ligh of certain guiding principles in a unified way. A decisive step in thi direction within the theory of entire and meromorphic functions consiste in replacing the classical representation of these functions through ca nonical products with more general tools from the potential theor (Green's formula and especially the Poisson-Jensen formula). On thi foundation it was possible to introduce the quantities (the characteristic the proximity and the counting functions) which are definitive for th description of the asymptotic properties of an analytic function in th vicinity of essential singularities. At the same time they lead to far reaching extensions and sharpenings of Picard's theorem, in the directior of the general value distribution theory, which is concerned with th distribution and density of those points at which an analytic functior assumes a preassigned value, whereby all complex values are to b considered. This change in method has also led to new insights in anothei direction: it has helped to bring the algebraic-analytic points of view basic to the Cauchy-Weierstrass function theory into closer contact witl the principles of the Riemannian function theory, which in greatei measure emphasizes the potential theoretical and geometrical features o the analytic mappings under study.

This new edition contains some changes and additions, particularly concerning the Second Main Theorem. At the suggestion of severa colleagues, I have included my "elementary" proof of the theorem. also give a version of F. NEVANLINNA's differential geometrical methoc which makes the main theorem easier of access.

In the course of the last three decades the literature on the topic treated here and related questions has grown enormously. Of the work which contain important contributions to these problems I wish to mention in particular:

WEYL, H.: Meromorphic Curves and Analytic Curves. Princeton: Princeton University Press 1943.

HAYMAN, W. K.: Meromorphic Functions. Oxford: Clarendon Press 1964.

WITTICH, H.: Neuere Untersuchungen über eindeutige analytische Funktionen. Berlin-Göttingen-Heidelberg: Springer 1955.

This book has been translated into English with great care and interest by Dr. P. EMIG. For this I express to him my sincere thanks. I am also indebted to Cand. Phil. T. HEIKKURINEN for her help in the reading and correction of the proofs. Finally, I should like to express my thanks to Springer-Verlag for their friendly cooperation in the production of this volume.

Helsinki, February 1970

ROLF NEVANLINNA

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Introduction

Analytic functions can be investigated from various points of view. The problems discussed in this book are grouped around one great fundamental question. Some general remarks concerning this central inquiry are offered here.

Let us imagine continuing a given analytic function element indefinitely. Assuming that the resulting analytic function $w = w(z)$ is single-valued, there exists a schlicht region G_z with the following properties.

1. To each interior point z of G_z there corresponds one and only one element of rational character of the function $w(z)$.

2. Each boundary point z^* of G_z is an essential singularity of $w(z)$.

If G_z spans the entire closed plane (elliptic case), then $w(z)$ is a rational function. If this simplest special case is excluded, there are two cases to be distinguished, according as G_z is simply or multiply connected. We restrict ourselves to the first case and then have two further possibilities to take into account: the boundary Γ_z of G_z is either one point (parabolic case) or a continuum (hyperbolic case).

The region G_z is conformally mapped by the function $w = w(z)$ onto a Riemann surface G_w over the w-plane. The inverse function $z = z(w)$ of $w(z)$ is single-valued and because of the single-valuedness of $w(z)$ *univalent* on this surface, i.e., the central points of two different function elements of $z(w)$ are always related to two different points z.

Conversely, according to the main theorem in the theory of conformal mapping, it is true that an arbitrary simply connected Riemann surface can always be related to one of the following three schlicht normal regions G_z one-to-one and conformally: 1. the extended plane (elliptic case); 2. the punctured plane (parabolic case); 3. the unit disk or, more generally, an arbitrary region bounded by a continuum Γ_z (hyperbolic case).

The value distribution theory of single-valued analytic functions is concerned with the investigation of the system (z_a) of those points in G_z where the function $w(z)$ assumes a previously given value $w = a$; here all values a are taken into consideration. The classical result in this area is the famous theorem of Picard, according to which a function which is meromorphic and transcendental in the punctured plane assumes all except for at most two values a infinitely often. This was the point of departure for the value distribution theories of HADAMARD, BOREL VALIRON, JULIA and others. In these far-reaching generalizations and refinements of Picard's theorem concepts from conformal mapping played a subordinate role. The problems were set up and treated with no regard for the Riemann surface G_w onto which the meromorphic function maps the punctured plane G_z. Conversely, the first investigations concerned with the Riemann surface of an entire or meromorphic function (HURWITZ, BOUTROUX, IVERSEN, GROSS and others) had but few points of contact with the results from the value distribution theory.

The modern theory of meromorphic functions has built a bridge between the two lines of research. In place of the Weierstrass canonical representation of an entire or meromorphic function appear function and potential theoretical tools of greater power. On the new foundation the Hadamard-Borel theory can be simply presented and sharpened. But the most important advance is that the new analytic concepts have at the same time a geometric meaning. The fundamental quantities (characteristic, deficiencies, ramification indices) relate the asymptotic behavior of a single-valued analytic function to the properties of the Riemann surface G_w onto which the schlicht existence domain G_z is conformally mapped. Picard's theorem and its generalizations can in this way be thought of as statements concerning the ramification character of the covering surface G_w. This new turn shifts the emphasis of the value distribution theory to an investigation of the distortion of the mapping $G_z \to G_w$ in the interior and above all in the vicinity of the boundaries of these surfaces; thus we are dealing with a far-reaching generalization of classical theorems on distortion and boundary correspondence under conformal mapping of schlicht regions.

The value distribution theory is thus integrated into the general theory of conformal mapping. From this point of view the central problem in the former theory is the *type problem*, an interesting and complicated question, left open by the classical uniformization theory. The problem is to decide based on the topological-metric properties of an open covering surface whether the surface is parabolic or hyperbolic, i.e., whether the surface can be mapped conformally onto the punctured plane or onto the interior of the unit disk. Those extensions of Picard's theorem customarily grouped under the name "deficiency relations" contain necessary criteria for the parabolic case. Although various sufficient conditions are also known, one is still not in possession of a complete solution of the type problem, even for simple classes of Riemann surfaces.

Among the Riemann surfaces of parabolic type those associated with a meromorphic mapping function $w(z)$ of finite order stand out in particular because of their simple ramification properties. To these belong, among others, the surfaces with finitely many branch points.

Among the hyperbolic surfaces certain noteworthy classes of surfaces are also singled out. A critical order of growth can be given for the characteristic of the mapping function above which the theorems of Picard and all of the deficiency relations are still valid. As the order of growth of the characteristic function decreases, the extent of the Riemann surface's ramification increases. Finally, the surfaces of *bounded type*, associated with a bounded characteristic, form a well-defined class. In order to be able to accurately describe the extent of the singularities of a surface of bounded type, the concept of a point set of capacity zero must be introduced. These point sets also play an important role for other questions in the value distribution theory. In our presentation the capacity notion is integrated into the theory of the so-called harmonic measure which I introduced in my lectures at the ETH in Zürich, during 1928-29. By the systematic use of this measure, which is invariant with respect to one-to-one conformal mappings, it becomes possible to treat various special questions closely related to the value distribution theory with unity and clarity.

In recent years knowledge concerning the type of a Riemann surface has been significantly extended through studies of abstractly defined Riemann surfaces of arbitrary connectivity or genus. Going into these general problems would lead outside the framework of this monograph, which is concerned with surfaces of genus zero. In the present new edition we must be content with a reference to the method of extremal lengths of GRÖTZSCH, AHLFORS and BEURLING and, in the case of complex manifolds of higher dimensions, to studies of STOLL, CHERN, BOTT and WU.

I. Conformal Mapping of Simply and Multiply Connected Regions

§ 1. Conformal Mapping by Means of Linear Transformations

1.1. The group of one-to-one conformal mappings of the extended plane onto itself is given analytically by the set of linear fractional transformations of the form

$$S(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0). \tag{1.1}$$

Every linear transformation produces such a mapping, and conversely every transformation $t(z)$ that carries out a mapping of this kind can be written in the form (1.1); for $t(z)$ is then regular at every point of the extended z-plane, with the exception of one single point z_0, which is mapped to $t = \infty$; consequently, the only singularity of $t(z)$ is a pole of order one, and according to an elementary theorem this implies that $t(z)$ is a rational function of order one.

1.2. If we now proceed by stereographic projection from the z-plane to the Riemann sphere of diameter 1 resting at the origin, we obtain from (1.1) the set of conformal mappings of the sphere onto itself. Among these mappings, the group formed by the rotations of the sphere merits special attention. The general form of a rotation that takes the point z to ζ and a to b is

$$\frac{z - a}{1 + \bar{a}z} = e^{i\alpha} \frac{\zeta - b}{1 + \bar{b}\zeta}, \tag{1.2}$$

where the bar denotes conjugation and α is a real parameter. By taking the limit as $z \to a$ and $\zeta \to b$ one sees that

$$\frac{|dz|}{1 + |z|^2} = \frac{|d\zeta|}{1 + |\zeta|^2},$$

from which it is evident that the expression

$$d\sigma = \frac{ds}{1 + r^2}, \tag{1.2'}$$

where ds is the euclidean arc length differential and r denotes the distance from the arc element in the z-plane to the origin, is invariant with respect to rotations of the sphere. This can be explained geometri-