Analytic Trigonometry

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PERGAMON PRESS
OXFORD · LONDON · PARIS · FRANKFURT

THE MACMILLAN COMPANY
NEW YORK

PERGAMON PRESS LTD.
Headington Hill Hall, Oxford
4 & 5 Fitzroy Square, London W.1

THE MACMILLAN COMPANY
60 Fifth Avenue, New York 11, New York

COLLIER-MACMILLAN CANADA, LTD.
132 Water Street South, Galt, Ontario, Canada

GAUTHIER-VILLARS ED.
55 Quai des Grands-Augustins, Paris 6

PERGAMON PRESS G.m.b.H.
Kaiserstrasse 75, Frankfurt am Main

Library of Congress Card No. 63-11608

Set in 10 on 12 pt. Times and Printed in Great Britain by
Charles Birchall & Sons, Liverpool and London

Preface

MUCH of the traditional matter of trigonometry is still fundamental in modern science and technology. A few of the basic elements of analytic geometry give to trigonometry a more modern flavor but, what is more important, a broader treatment of the trigonometric functions is made possible. Modern trigonometry is analytic trigonometry.

A recent study by the Commission of the College Entrance Examination Board envision trigonometry "centered around coordinates, vectors, and complex numbers". The report does not say how this is to be accomplished. Certainly, basic theories of learning and pedagogy must not be ignored.

The report of the 1961 Southampton Mathematical Conference states, "Mathematics should be seen as a developing subject; where appropriate, the pupil should be encouraged to see his work in relation to what has gone before, and to its possible significance in the future." It also reminds that, "Many students find that it is extremely difficult to bridge the gap which at present exists between school and university mathematics, both in content and presentation."

Both of the above reports have been considered in the presentation of the material of analytic trigonometry. The trigonometric functions are introduced in a rectangular coordinate setting, first as functions of angles and then extended to functions of real numbers. The polar coordinate system is introduced next to enable the student to see the trigonometric functions in another important context. Finally, the introduction of complex numbers and series add breadth by opening up some of the many areas of application of the trigonometric functions. The importance of teaching for transfer is recognized throughout, particularly in bridging the gap between elementary algebra and trigonometry. The presentation progresses from the easier to the more difficult in topical order, text, and problems. Certainly this is pedagogically sound.

All definitions are consistent with modern standards of rigor. Special care has been taken so as to be mathematically correct throughout.

Chapter I introduces some of the essential elements from elementary algebra, plane geometry, and analytic geometry. Chapters II and III present definitions of the trigonometric functions of angles in general, basic identities, and solutions of equations. Trigonometric functions of real numbers are treated in Chapter IV. A one to one identical correspondence between real numbers and measures of angles in radians is established through correspondence of measures of arcs and angles in radians in a unit circle. Real number solutions of equations, including approximation methods, are presented. Chapter V defines inverse trigonometric functions as single-valued functions from the outset. Identities, equations, and methods of expressing general solutions of trigonometric equations are included. A thorough treatment of procedures for sketching graphs of trigonometric functions with analogies from algebra comprises Chapter VI. Chapters VII and VIII cover logarithms and adapt the definitions of the trigonometric functions to the solutions of triangles. Vectors are introduced at this point for use in the problems and because they are needed later in the material on polar coordinates and complex numbers. Applications of the Law of Sines and the Law of Cosines are stressed. The polar coordinate system is introduced in Chapter IX. Here the polar distance is defined and used as a positive number. Chapter X treats complex numbers and series representation. The polar form of a complex number and its vector representation are stressed. Series approximations are included.

The inclusion of a supplement by another author is a departure from the usual. It is hoped that Analysis of the Definitions of the Trigonometric Functions by Dr. E. L. Whitney will serve as enrichment material.

Answers to all problems are included to encourage students to attempt a larger number of items.

Much of the treatment of the material of the text is based on the author's own many years of experience in teaching trigonometry both in the United States and in Canada. Invaluable assistance has come from reviewers of the original manuscript and from the comments and suggestions of others. The author wishes to express his sincere appreciation to his colleagues Professors G. C. Cree, E. S. Keeping, E. Phibbs, J. R. Pounder, and E. L. Whitney whose contributions were most helpful.

To the publishers, Pergamon Press, who made every effort to offer encouragement and assistance from the outset, the author is grateful.

Without the patience and tolerance of the Bruces, Marjorie, Linda, and Barbara, over the many months of preparation, this book could not have been completed.

Edmonton, Alberta WM. J. BRUCE

GREEK ALPHABET

Alpha	A, α	Nu	N, ν
Beta	B, β	Xi	Ξ, ξ
Gamma	Γ, γ	Omicron	O, o
Delta	Δ, δ	Pi	Π, π
Epsilon	E, ϵ	Rho	P, ρ
Zeta	Z, ζ	Sigma	Σ, σ, ς
Eta	H, η	Tau	T, τ
Theta	Θ, θ	Upsilon	Υ, υ
Iota	I, ι	Phi	Φ, φ
Kappa	K, κ	Chi	X, χ
Lambda	Λ, λ	Psi	Ψ, ψ
Mu	M, μ	Omega	Ω, ω

Bruce: ANALYTIC TRIGONOMETRY

Errata

p. 3 Line 8 from bottom. $x < -2$ should be $x > -2$.

p. 15 Line 9. $\frac{4\pi}{3}$ should be $\frac{7\pi}{6}$. $\frac{7\pi}{3}$ should be $\frac{13\pi}{6}$.

Line 13. $\frac{4\pi}{3}$ should be $\frac{7\pi}{6}$.

pp. 19, 20 and 21. The curves in Figs. 2.7, 2.8, 2.9 and 2.10 should not touch the vertical asymptotes.

p. 27 Question 14. $x = \sin\theta$ should be $x = 2\sin\theta$.

p. 31 Question 21. $2\cos\theta$ should be $2\csc\theta$.

p. 36 Line 5. $\cos\theta = \frac{3}{5}$ should be $\cos\theta = -\frac{3}{5}$.

p. 41 Line 6. $\frac{1}{2}$ should be $\pm\frac{1}{2}$.

p. 45 Question 10 should be $\tan\theta = -\frac{1}{3}\sqrt{3}$.

p. 50 Line 5 from bottom. $\mathbf{tan}\,(\mathbf{90}^\circ - d)$ should be $\mathbf{tan}\,(\mathbf{90}^\circ - \beta)$.

p. 52 Example 2. Lines 4, 5, 6 and 7 of this example should read "cos β. Since $\alpha > \beta$, α must be in quadrant II and β in quadrant I or II.

Thus, $\cos\alpha = -\frac{4}{5}$ and $\cos\beta = \pm\frac{5}{13}$.

$$\text{Then, } \sin(\alpha - \beta) = \frac{3}{5}\left(\pm\frac{5}{13}\right) - \left(-\frac{4}{5}\right)\frac{12}{13}$$
$$= \frac{33}{65} \text{ or } \frac{63}{65}.\text{"}$$

p. 54 Line 8 from bottom. $\cos^2\alpha$ should be $\cos 2\alpha$.

p. 57 Question 15. $\cot\theta$ should be $\tan\theta$.

p. 58 Line 18 should be
"and $\cos(x + y) - \cos(x - y) = -2\sin x \sin y$ (4)"

p. 60 Question 9 (b) $0 \leq \theta < 2\pi$.

p. 61 Fig. 3.5. Curve is drawn "too fat". Shape should be like that of the curve in Fig. 3.6.

p. 75 Line 20. $y = \text{Arcsin}\, x$ should be $y = \text{arcSin}\, x$.

Errata (*continued*)

p. 78 Line 2. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Line 3. "such that $\theta' = -\theta$".

p. 82 Line 16. θ = Arcsin x should be θ = arcsin x.

p. 85 Line 3 from bottom. "solution" should be "solutions".

p. 90 Line 3 from bottom. Change to read, "Since $\tan(\theta + \pi) = \tan\theta$, and since π is the smallest positive number for which this is true, the period of $\tan\theta$ is π."

p. 96 Question 16. $y = \cos\frac{1}{2}\theta$.

p. 99 Line 4. "$y = \sin\theta$ is moved to the left $\frac{\pi}{4}$ units".

Fig. 6.10. Graph $y = \sin\left(\theta + \frac{\pi}{4}\right)$ is the one to the left of the one labelled.

p. 104 Section 6.5. Change to read

"Let us consider functions of the type

$$A\cos\theta \pm B\sin\theta,\ A > 0\text{''}.$$

p. 105 Insert just ahead of Example 1 the following:
"Note: If $A < 0$, other than the principal value of λ will have to be chosen by considering the signs of $\cos\lambda$ and $\sin\lambda$."

p. 108 Line 5 should read, "graph of $\sin b\theta$ and **not** of $\sin\theta$".

p. 122 Line 3. 1.95566 should be 1.93566.

p. 134 Line 15. **bound vector** should be **bound vectors**.

p. 142 Line 3 from bottom should read
"$\gamma = 180^\circ - (33^\circ\,39' + 46^\circ\,47') = 99^\circ\,34'$."

p. 143 Line 4. Add in parentheses leaving some space
"($\sin 99^\circ\,34' = \sin 80^\circ\,26'$)".

p. 214 Paragraph 2 line 3. $\sin\alpha\sin\beta$ should be $\sin\alpha + \sin\beta$.
Section 23 line 7 should be
"Area $\triangle OXP = \ldots =$ Area $\triangle OPT$."

p. 223 Exercise 2.2 Question 16. $\cot^3\theta$ should be $\tan^3\theta$.

p. 226 Exercise 3.4 Question 3(a). $-.80316$ should be $-.72617$.
Exercise 4.1 Question 17. .9272 should be .4041.
Exercise 4.2 Question 11. *Omit* $x = 0$, $y = 0$.

I Foundations

ANALYTIC TRIGONOMETRY is a study of trigonometry using the methods of analytic geometry whenever appropriate to do so. The word trigonometry comes from the Greek word **trigōnon** meaning triangle and **metria** meaning measure and was used by Hipparchus in the second century B.C. Ptolemy in the second century A.D. extended the subject using geometry, but it was not until the sixteenth century that Vieta, a Frenchman, employed algebra in the study of trigonometry. The eighteenth-century work of De Moivre and Euler and the nineteenth-century contributions of Cauchy followed by the discovery of the periodic nature of the trigonometric functions shifted the emphasis to a study of the trigonometric functions far removed from that of triangle solving. It is the properties of these trigonometric functions which makes them widely applicable in mathematics, physics, and engineering.

René Descartes, in the seventeenth century, discovered a connection between algebra and geometry by associating the numbers of algebra with the points of geometry. Thus was born analytic geometry, the methods of which are frequently used today is the study of trigonometry. It seems appropriate, therefore, to call our subject matter analytic trigonometry.

Certain essential elements of algebra, geometry, and analytic geometry are basic to our study and are included herein.

1.1. Essential Elements from Algebra

As in algebra, it will be necessary in our study of trigonometry to distinguish equations from identities. Let us recall that an **equation** is an equality that is true only for certain values of the unknown. The equality $2x - 6 = 0$ is true if and only if $x = 3$, whereas the equality $x^2 - x - 6 = 0$ is true if and only if $x = -2$ or 3. Both of these equalities are examples of algebraic equations. An **identity** is an equality that is true for all values of the unknown for which both members are defined. The equality $\frac{x^2 - 4}{x - 2} = x + 2$ is true for all values of x except $x = 2$ and is an algebraic identity.

A common method of proving that an equality is an identity is that of showing that one member transforms exactly into the other, subject possibly to certain restrictions. In the above example, we have

$$\begin{aligned}\frac{x^2 - 4}{x - 2} &= \frac{(x - 2)(x + 2)}{x - 2}\\ &= x + 2, \text{ provided } x \neq 2\end{aligned}$$

and we conclude that

$$\frac{x^2 - 4}{x - 2} = x + 2, x \neq 2$$

In identities, it is sometimes customary to use the symbol $\equiv$ (meaning identically equal to) instead of the symbol $=$, particularly when no provisos are involved.

One of the most important equations is the **quadratic equation** $ax^2 + bx + c = 0$, $a \neq 0$. Here a, b, and c are constants. Two methods of solution have been seen in elementary algebra. By factoring the quadratic, an equivalent equation of the form $a(x - r_1)(x - r_2) = 0$, where r_1 and r_2 are constants, results. This equality is true if and only if either $x - r_1 = 0$ or $x - r_2 = 0$. The roots of the quadratic equation are thus r_1 and r_2.

Example 1. Solve $2x^2 + 5x - 3 = 0$ by factoring.

Factoring the left member, we obtain the equivalent equation

$$(2x - 1)(x + 3) = 0$$

or $$2(x - 1/2)(x + 3) = 0$$
This is true if and only if either $x - 1/2 = 0$ or $x + 3 = 0$, from which we obtain $x = 1/2$ and $x = -3$.

When the factors of the quadratic are not obvious or when the roots of the equation are not rational numbers (numbers of the form $\frac{m}{n}$, $n \neq 0$, where m and n are integers) it was seen in elementary algebra that the solutions of $ax^2 + bx + c = 0$, if they exist, are given by
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
The formula in the right member is called the **quadratic formula** and is obtained by transforming the quadratic into the form of a perfect square and then extracting the square roots. Whether the roots are real or not depends on whether $b^2 - 4ac$ is non-negative or negative.

Example 2. Solve $2x^2 - 4x - 3 = 0$ by the quadratic formula.

Applying the quadratic formula with $a = 2$, $b = -4$, and $c = -3$, we obtain $x = \frac{4 \pm \sqrt{16 + 24}}{4} = \frac{4 \pm \sqrt{40}}{4} = \frac{2 \pm \sqrt{10}}{2}$.

Also, we shall have occasion to use the standard **inequality symbols.** The symbol $\neq$ (not equal to) was used earlier. When we write $x > 3$ we mean the set of all numbers greater than the number 3, whereas $x < 3$ means the set of all numbers less than 3. The statement $x < -2$ means the set of all numbers greater than -2. When we wish to include -2 in this set, we write $x \geq -2$, read "x greater than or equal to -2". The expression $-2 < y < 3$ is read "y greater than -2 and less than 3".

The **absolute value** of a number is the corresponding numerical value. The absolute value of 2 is thus 2 and the absolute value of -2 is also 2. In symbols, we write $|-2| = 2$. Note that the absolute value is always a non-negative number. Thus $|a| = a$,

$a \geq 0$, whereas, $|a| = -a$, $a < 0$. We take this as our definition of $|a|$.

It is necessary to give a unique meaning to the **square root symbol** $\sqrt{\ }$. By this symbol we agree to mean the positive square root. Thus $\sqrt{4} = 2$. If we desire the negative square root, we must write $-\sqrt{4}$ to obtain -2. This must not be confused with the solution of $x^2 = 4$ which yields $x = \pm\sqrt{4} = \pm 2$.

For our purpose, we shall need a precise definition of a function of one variable. If two variables x and y are so related that corresponding to each admissible value of x there is one and only one value of y, we shall say that **y is a function of x.** The values of x form a set of numbers called the **domain** of the function; the corresponding values of the function y form a set of numbers called the **range** of the function. In this case, x is called the **independent variable** and y the **dependent variable** or **function.** The symbol $f(x)$, read "f of x", is commonly used for y and we write $y = f(x)$. However, $f(x)$ also means the number, specified by the function, that corresponds to x. Note that our definition of a function insists that the function be single-valued. We shall be interested in functions which can be represented by formulas, however, our definition does not require that a formula be necessary.

Example 3. If $y = f(x) = x^2$, the domain is "all x", whereas the range is $y \geq 0$.

Example 4. If $y = \sqrt{1 - x^2}$, the domain is $-1 \leq x \leq 1$, whereas the range is $y \geq 0$. Note that the domain may also be written $|x| \leq 1$.

The concept of an **inverse function** is also needed in our study of trigonometry. Consider any function $y = f(x)$ satisfying our definition of a function and such that $x = g(y)$ also satisfies this definition. Let us proceed as follows:

(1) Interchange the roles of x and y in $y = f(x)$.

(2) Solve for y in terms of x (This will not be always possible).
(3) Specify the domain of this new function to correspond to the range of the given function $f(x)$. (The range of our new function will correspond to the domain of the given function).

A function determined in this manner is said to be the **inverse** of the given function.

Example 5. Find the inverse of $y = \sqrt{x}, x \geq 0$.

We follow the above procedure. Interchange x and y to obtain $x = \sqrt{y}$. Solve for y and get $y = x^2$. The range of $y = \sqrt{x}$ is $y \geq 0$, hence the domain of our new function is $x \geq 0$. Therefore, $y = x^2, x \geq 0$, is the required inverse.

Example 6. Show that $y = x^2$ does not have a unique inverse.

If we proceed as above, we get $x = y^2$ upon interchanging x and y, and $y = \pm \sqrt{x}$ when we solve for y. But $y = \pm \sqrt{x}$ does not define a function according to our definition. (Here y is not

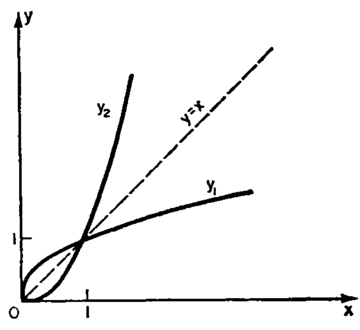

FIG. 1.1

single-valued.) Thus $y = x^2$ does not have an inverse. We could have shown this also by observing that $x = \pm \sqrt{y}$ does not define x as a function of y. (See the requirement on $x = g(y)$ given above.)

We now consider the graphical relation that exists between a function and its inverse. It was seen in Example 5 above that $y = \sqrt{x}$, $x \geq 0$ has as its inverse $y = x^2$, $x \geq 0$. The graphs of these are shown in Fig. 1.1. as y_1 and y_2, respectively. We note that y_2 is the mirror image of y_1 with $y = x$ as the mirror. Our method of obtaining an inverse shows that this is always the case, hence we have a very simple method for sketching the inverse of any function. Furthermore, even though, as is pointed out in step (2) of our definition, we are unable to obtain y in terms of x, it is still possible to give a graphical representation of the inverse merely by interchanging the x and y coordinates of the points of the graph of the function.

Exercise 1.1

1. Solve by factoring:
   (a) $x^2 - 5x + 6 = 0$ (b) $2x^2 - 5x + 3 = 0$
2. Solve by using the quadratic formula:
   (a) $3x^2 + 5x - 7 = 0$ (b) $x^2 - 2x - 5 = 0$
3. Solve: $\dfrac{x - 1}{x^2 - 4x + 4} = \dfrac{1}{2} + \dfrac{3x}{x - 2}$
4. Solve: $\sqrt{3x + 1} + x = 3$. Check your solution. Explain why one of the values obtained is not a solution. Of what is it a solution?
5. Prove: $\dfrac{(x^2 - 4)(x + 3)}{x^2 + x - 6} = x + 2, x \neq 2, x \neq -3.$
6. Solve: $|x - 1| = 3$
7. Express $|x| < 3$ as an inequality without the absolute value symbol.
8. Determine $|a|^2$; $|a - b|^2$.
9. What is the domain of the function $y = \sqrt{1 - x}$? What is the range?
10. Determine the inverse of each of the following and sketch the graphs:
   (a) $y = x^3$ (b) $y = x^4, x \geq 0$
   (c) $y = x^4, x \leq 0$ (d) $y = x^2 - 1, x \geq 0$
   (e) $y = x^2 + 1, x \leq 0$

1.2. Essential Elements from Plane Geometry

A theorem widely used in trigonometry is the one attributed to

Pythagoras, namely, the square on the hypotenuse of a right triangle is identically equal to the sum of the squares on the two sides. If a and b are the lengths of the legs and c the length of th hypotenuse of a right triangle, then $c^2 = a^2 + b^2$.

The concept of an angle and its measure in degrees, minutes, and seconds as given in plane geometry is extended in trigonometry. A new measure of an angle will be used as follows. Consider a circle of any radius r and with central angle θ such that the radii intercept an arc S on the circumference equal in length to the radius of the circle. The angle θ is then said to be an angle of one **radian** and has a measure of one radian (Fig. 1.2). If the circle is a unit circle (one of radius equal to one or unity) we note that all of S, r, and θ have one unit of measure in this case. Thus the central angle has a measure given by the length of arc S.

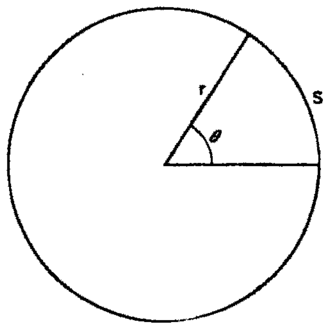

FIG. 1.2

We shall assume that the circumference of a circle is given by $C = 2\pi r$. Then, since r divides into $2\pi r$ exactly 2π times, we find that a radius turned through one complete revolution generates an angle of 2π radians at the center of the circle. Thus 2π radians are equivalent to 360° or π radians are equivalent to 180° and 1 radian is equivalent to $\dfrac{180°}{\pi}$ which is about 57.3°. On the other hand, 1° is equivalent to $\frac{\pi}{180}$ radians which is about 0.01745 radians. It is clear that the unit of radian measure is a much larger unit than that of degree measure.

We note that for any circle the proportion

$$\frac{S}{2\pi r} = \frac{\theta}{2\pi}$$

holds and from this we obtain $S = r\,\theta$, where θ is in radian measure. It follows that $\theta = \frac{S}{r}$ and, since S and r will be in the same linear units, we find that θ will be void of units and is hence called a pure number.

Example 1. Express 270° in radian measure.

Since 1° is equivalent to $\frac{\pi}{180}$ radians, we have that 270° are equivalent to $270 \times \frac{\pi}{180} = \frac{3\pi}{2}$ radians.

Example 2. Find the central angle in radians for a circle with $r =$ 2 ft and $S = 3$ ft.

We have immediately $\theta = \frac{S}{r} = \frac{3 \text{ ft}}{2 \text{ ft}} = 1.5$ (radians).

Exercise 1.2

1. Find the angle in radians subtended at the center of a circle of radius 4 ft by an arc of 3 ft.
2. Change to radians:
   (a) 45° (b) 10° (c) 135° (d) 22° 15′
3. Change to degrees the following angles given in radians (use 1 radian = 57.2958° where needed):
   (a) $\frac{\pi}{3}$ (b) $\frac{5\pi}{4}$ (c) 1.6 (d) 2.3
4. A wheel makes 30 rev/min. Express this rate in radians per minute. How long does the wheel take to turn through π radians?

5. Assume the radius of the earth to be 4000 miles and that the earth is roughly a sphere. Find the length of arc on the circumference intercepted by a central angle of 1°.
6. Assume that the area of a circle is given by πr^2 and show that the area K of a sector of a circle of radius r and central angle θ is given by $K = \frac{1}{2} r^2 \theta$, where θ is in radian measure. Show that this area is also given by $K = \frac{1}{2} rS$, where S is the arc length.
7. Find the area of a circular sector of radius 2 ft and central angle 2.5.
8. For small angles a circular arc is approximately equal to the chord which it subtends. Use this fact to determine the approximate diameter of the sun if the angle subtended by it at the eye of an observer is 32′. Assume the sun to be 92,500,000 miles distant.

1.3. Essential Elements from Analytic Geometry

Consider the real numbers as associated with points on a line. Such a line is called a **coordinate line** and the numbers associated with the points are called coordinates of the points.

P	Q	R	S	O	A	B	C	D
-4	-3	-2	-1	0	1	2	3	4

FIG. 1.3

We assign O to the number 0, A to the number 1, and so on as shown in Fig. 1.3. We call O the **origin** and the length from O to A the **scale unit.**

Let p be the number assigned to the point P on this line. We shall use the notation $P(p)$ to mean the point P with coordinate p. If $Q(q)$ is another point on the line, we define the **length** of the segment PQ as $|PQ| = |q - p|$. This is the **distance** between P and Q and is always a positive number by this definition.

Example 1. Find the distance between $P(5)$ and $Q(-3)$ on a coordinate line.

Solution: The required distance is given by

$$\begin{aligned} |PQ| &= |-3-5| \\ &= |-8| \\ &= 8 \end{aligned}$$

On the other hand, we define the **directed distance** from the point $P(p)$ to the point $Q(q)$ on a coordinate line as

$$PQ = q - p$$

Since $q - p$ may be positive or negative, we see that direction is involved and hence the term "directed distance". It follows that

$$QP = p - q$$

and thus that $PQ = -QP$

or $PQ + QP = O$

Also, we note that $|PQ| = |QP|$

Now, if $Q(q)$ is to the right of $P(p)$, $q > p$ and $PQ = (q - p) > O$. On the other hand, if $Q(q)$ is to the left of $P(p)$, $q < p$ and $PQ = (q - p) < O$.

Example 2. Find the directed distance from $P(5)$ to $Q(-3)$ on a coordinate line.

Solution: The directed distance is given by

$$\begin{aligned} PQ &= -3-5 \\ &= -8 \end{aligned}$$

We have just shown how a coordinate system can be defined on a line. Let us now describe a coordinate system for a plane. We use a pair of coordinate lines perpendicular at their origins and having equal scale units. These lines we call the **coordinate axes.** One of the lines is usually placed in a horizontal position and is often called the "x — axis" while the other is then called the "y — axis". In this system, we associate with each point P an ordered pair of numbers. If a and b are a pair of such numbers we shall write $P(a, b)$ to indicate point P with directed distance a from the y — axis and directed distance b from the x — axis. We call a and b the coordinates of the point P, a is the **x-coordinate** (or abscissa) and b the **y-coordinate** (or ordinate). This unique association of pairs of numbers with points in a plane is called a **rectangular coordinate system** or a Cartesian coordinate system after Descartes.

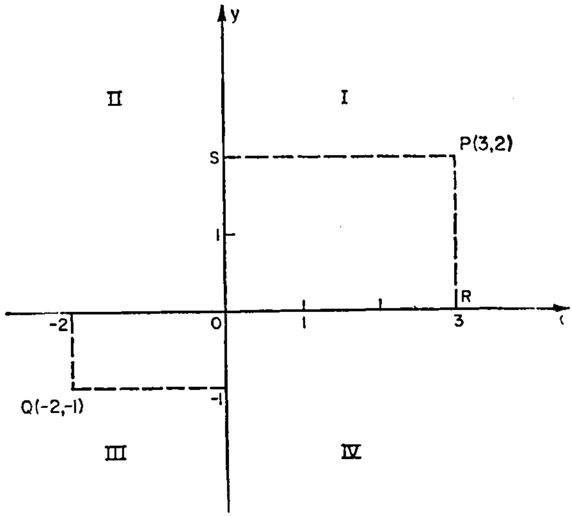

FIG. 1.4

The coordinate axes divide the plane into four **quadrants,** numbered as in Fig. 1.4. In quadrant I the coordinates are both positive, in quadrant II $a < 0$ and $b > 0$, and so on.

Corresponding to the point P (a, b) is a point R (a) on the x-axis (a coordinate line). Such a point R (a) is called the **projection** of the point P (a, b) on the x-axis. Similarly, the point S (b) is the projection of P (a, b) on the y-axis (Fig. 1.4). Note that the point R (a) on the x-axis is also the point R $(a, 0)$ in the plane and that the point S (b) is the point S $(0, b)$. The origin has coordinates (0, 0).

We now consider the distance between any two points in rectangular coordinates.

THEOREM: *If P_1 (x_1, y_1) and P_2 (x_2, y_2) are any two points, then the distance d between P_1 and P_2 is given by*

$$d = |P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: From Fig. 1.5 and the Pythagorean theorem

$$|P_1 P_2|^2 = |P_1 T|^2 + |T P_2|^2$$

Now $S R$ is the projection of $P_1 T$ on the x-axis, hence $|P_1 T| = |S R|$. But S and R have x-coordinates x_1 and x_2, respectively.

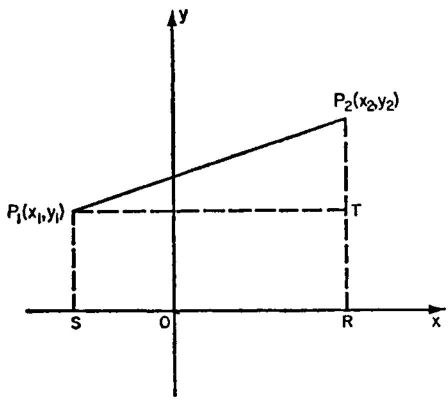

FIG. 1.5

Therefore,

$$|S R| = |x_2 - x_1| = |P_1 T|$$

Similarly, by projecting $P_1 P_2$ on the y-axis, we find

$$|T P_2| = |y_2 - y_1|$$

Hence $\quad |P_1 P_2|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and $\quad d = |P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The method of proof used above is called the analytic method.

Example 3. Find the distance between $(-2, 5)$ and $(4, -3)$.

$d = \sqrt{(4 - [-2])^2 + (-3 - 5)^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$